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OPTIMALITY CONDITIONS IN THE PROBLEM OF SEEKING THE HOLE SHAPES IN ELASTIC BODIES

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Seeking the optimal hole shapes in elastic bodies, which cause minimal stress concentration, results in minimax optimization problems with a local criterion. These problems are considered in this paper within the framework of plane elasticity theory, and it is proved that holes with equi-stressed boundaries are optimal.

Problems of boundary optimization were investigated earlier in [1-4] in connection with seeking the shapes of twisted rods possessing maximum torsional stiffness. and problems to construct equi-stressed holes in plates were studied in [5-8].

1. Let us examine the plane problem of elasticity theory on the state of stress of an infinite plate weakened by a hole. Let G denote a domain in the xy plane which is occupied by the plate material, while Γ is the hole boundary. Let us assume that the plate stretches to infinity, while the hole contour Γ is free of applied loads. We write the appropriate boundary conditions on Γ and the condition at infinity in the form

$$\sigma_n = 0, \quad \tau_n = 0 \quad (x, y) \in \Gamma \tag{1.1}$$
$$(\sigma_x)_\infty = \sigma_1, \quad (\sigma_y)_\infty = \sigma_2, \quad (\tau_{xy})_\infty = 0$$

where σ_1 and σ_2 are given positive constants, and n and t denote the normal and tangent directions to the contour.

For a given contour shape Γ the stress distribution in the domain $G + \Gamma$ is determined completely by conditions (1.1). As is known, using the stress function φ associated with the stress tensor components by means of the relationships $\sigma_x = \varphi_{yy}$, $\sigma_y = \varphi_{xx}$, $\tau_{xy} = -\varphi_{xy}$, reduces the problem of seeking the stresses to the solution of the biharmonic equation

$$\varphi_{xxxx} + 2\varphi_{xxyy} + \varphi_{yyyy} = 0 \quad (x, y) \in G \tag{1.2}$$

under the boundary conditions (1.1). The subscripts in (1.2) denote differentation with respect to the appropriate variables.

For each point $(x, y) \in G + \Gamma$ we characterize the state of stress by the function F of stress tensor invariants

$$F = F(I_1, I_2) \tag{1.3}$$
$$I_1 = \sigma_x + \sigma_y, \quad I_2 = \tau_{xy}^2 - \sigma_x\sigma_y$$

We understand F to be a function whose attainment at the point $(x, y) \in G + \Gamma$ of a given value k^2 (the constant k^2 is a material characteristic) means that the material is in the limit state at the point mentioned. Deformations of the material are elastic if the inequality $F < k^2$ is satisfied in appropriate domains. Violation of inequality is treated in different mechanics theories as the appearance of a flow zone, domains of inelastic strains, and a discontinuity in the continuity of the material and other effects. Henceforth, we shall interpret the condition $F = k^2$ as a plasticity condition. Giving F by the expressions

$$F = I_1^2 + 3I_2 \tag{1.4}$$

$$F = I_1^2 + 4I_2 \tag{1.5}$$

corresponds to the Mises and Tresca plasticity criteria. In addition to the two expressions mentioned for F, even more general dependences of F on the invariants I_1 and I_2 will be examined below. It will hence be assumed that the expression for F, which is represented in terms of the stress tensor components, is a homogeneous function with a homogeneity index of 2.

For a given Γ let the boundary value problem (1.1), (1.2) be solved for certain sufficiently small values of the parameters σ_1, σ_2 and let the stresses $\sigma_x(x, y)$, $\sigma_y(x, y)$, $\tau_{xy}(x, y)$ be thereby found. Then, a set of points $G_0 \subset G + \Gamma$ can be determined where the maximum of the function F

$$F_0 = (F)_{G_0} = \max_{xy} F, \quad (x,y) \in G + \Gamma$$

is realized.

Then if the values $(\sigma_x)_\infty$ and $(\sigma_y)_\infty$ increase in proportion to a certain parameter p, i.e., it is considered that $(\sigma_x)_\infty = p\sigma_1$, $(\sigma_y)_\infty = p\sigma_2$, then the plastic state will first be achieved at the points $(x, y) \in G_0$ for the value $p_0 = k / \sqrt{F_0}$ of this parameter. Evidently, the smaller the value of F_0, the more the plastic strains appear in the plate for large loads (large values of p_0). Hence, expansion of the range of loads for which the strains are elastic and fluidity zones do not occur in the plate is achieved because of minimizing the quantity F_0.

We arrive at the following optimization problem. Determine the shape of the contour Γ for which the minimum of the maximum is achieved for the quantity F in the domain $G + \Gamma$, i.e.,

$$F_* = \min_\Gamma F_0 = \min_\Gamma \max_{xy} F \tag{1.6}$$

The problem (1.1)-(1.3), (1.6) formulated is among the optimization problems with local quality criteria because of the local nature of the functional to be optimized. In seeking the minimum in Γ it is assumed in (1.6.) that the desired contour cannot shrink to a point, i.e., the absence of a cavity is not allowed. The shape of the hole contour plays the part of the "control" function, and (1.2) enters the optimization problem as a differential constraint.

2. Before investigating the optimality conditions in the problem formulated, let us note a property of harmonic functions which is used later.

In the xy plane, let there be n holes bounded by the closed contours Γ_k $(k = 1, 2, \ldots, n)$. Let us consider a family of harmonic functions, continuous in $G + \Gamma$ and tending to a given positive constant A at infinity, in a domain G bounded by the contour $\Gamma = \Sigma\Gamma_k$. For any function from this family, and taking account of its values on the boundary Γ $((\omega)_\Gamma = f)$ according to the maximum principle (see [9], for instance), the following inequality holds

$$|\omega(x, y)| \leqslant \max_{\xi\eta} |f(\xi, \eta)|, \quad (x, y) \in G + \Gamma, \quad (\xi, \eta) \in \Gamma$$

In particular, it hence follows that

$$A \leqslant \max_{\xi\eta} |f| \tag{2.1}$$

If the strict equality is realized in (2.1), then the function ω is identically equal to the constant A. Hence, the minimum of the functional $\max_{\xi\eta}|f|$ $((\xi, \eta) \in \Gamma)$ with respect to f is achieved on the unique function $f(\xi, \eta) \equiv A$ $(\omega(x, y) \equiv A)$ of the family under consideration, and its equals A, i.e.,

$$\min_f \max_{\xi\eta} |f| = A \tag{2.2}$$

We use the mentioned property of harminic functions below to estimate the stresses on the hole boundaries.

3. Let us first examine the expression (1.4) as F, which equals the square of the tangential stress intensity to the accuracy of a factor and corresponds th the Mises plasticity criterion.

Let us introduce the auxiliary function

$$\omega = \sigma_x + \sigma_y \tag{3.1}$$

which is harmonic, as is known (see [10], for example). Taking account of (1.1) and the equality $\sigma_x + \sigma_y = \sigma_n + \sigma_t$, we have

$$\Delta\omega = 0, \quad (x, y) \in G + \Gamma, \quad (\omega)_\Gamma = \sigma_t, \quad (\omega)_\infty = \sigma_1 + \sigma_2 \tag{3.2}$$

where Δ is the Laplace operator. Furthermore, using the invariance of the expression (1.4) relative to passage from the xy axes to the directions n and t, and the boundary conditions (1.1), we arrive at the following formula for the boundary values of F:

$$(F)_\Gamma = \sigma_t^2 \tag{3.3}$$

Applying the property of harmonic functions (2.2) noted to the function ω, defined by the relations (3.2) and comparing the expressions (3.2) and (3.3) for the boundary values of F and ω, we obtain that the minimum of the maximum values of $|\omega|$ and F on the contour Γ is achieved if and only if the stress σ_t is constant along the contour

$$(\sigma_t)_\Gamma = \sigma_1 + \sigma_2 \tag{3.4}$$

It is shown in [5] (see [6] also) that the contours Γ satisfying condition (3.4) form a one-parameter family of ellipses $x^2\sigma_1^{-2} + y^2\sigma_2^{-2} = \lambda^2$, where λ^2 is a parameter.

Now, let us show that compliance with the equality (3.4) is a necessary and sufficient condition for optimality in the problem (1.1) (1.3), (1.6).

For this it will be sufficient to prove that for the contours Γ satisfying condition (3.4) the maximum of the function F considered in the domain $G + \Gamma$ is reached on the contour Γ and hence

$$\max_{(x, y)\in G+\Gamma} F = \max_{(x, y)\in\Gamma} F \tag{3.5}$$

Indeed, from the equality (3.5) assumed and the fact that for arbitrary contours Γ the maximum of F in the domain $G + \Gamma$ is not less than the maximum of F on Γ it follows that the necessary and sufficient condition (3.4) for minimality of the maximum value of F on Γ will also be a necessary and sufficient condition for minimality of the maximum value of F in the domain $G + \Gamma$, where $F_* = \min_\Gamma \max_{(x, y)\in\Gamma} F$.

Let us use the complex representation of the stress tensor components in the terms of the potentials $\Phi(z)$, $\Psi(z)$ of Kolosov-Muskhelishvili ([10])

$$\sigma_x + \sigma_y = 4 \operatorname{Re} \Phi(z), \quad \sigma_x - \sigma_y + 2i\tau_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] \quad (3.6)$$

$z = x + iy$, $\bar{z} = x - iy$

Upon compliance with condition (3.4) the function $\omega = \sigma_x + \sigma_y = \sigma_1 + \sigma_2$ and therefore, $\Phi'(z) = 0$. The equations (3.6) are converted into [6]

$$\sigma_x + \sigma_y = \sigma_1 + \sigma_2, \quad \sigma_x - \sigma_y + 2i\tau_{xy} = 2\Psi \quad (3.7)$$

We multiply the second equation in (3.7) by the complex-conjugate. Taking account of (3.7), we will have

$$Q \equiv (\sigma_1 + \sigma_2)^2 + 4(\tau_{xy}^2 - \sigma_x\sigma_y) = 4\Psi\overline{\Psi} \quad (3.8)$$

Let us note that the function Q differs from the expression for F only by the factor in the second member. We express F in terms of Ψ and $\overline{\Psi}$. To do this, let us use (1.4) and (3.8). We have

$$F = {}^3/_4 Q + {}^1/_4 (\sigma_1 + \sigma_2)^2 = 3\Psi\overline{\Psi} + {}^1/_4 (\sigma_1 + \sigma_2)^2 \quad (3.9)$$

Furthermore, let us represent the functions Ψ and $\overline{\Psi}$ in the form $\Psi = u + iv$, $\overline{\Psi} = u - iv$, where the quantities u and v satisfy the Cauchy- Riemann conditions. Then (3.9) has the form $F = 3(u^2 + v^2) + {}^1/_4 (\sigma_1 + \sigma_2)^2$. Let us apply the Laplace operator to the expression obtained. Performing elementary manipulations and taking account of the Cauchy-Riemann condition as well as the resulting equality $\Delta u = \Delta v = 0$, it is easy to show that $\Delta F = 12(\nabla u)^2 \geqslant 0$. Therefore, F is a superharmonic function which does not achieve the maximum value at internal points of the domain $G + \Gamma$. Comparing the values $(F)_\Gamma = (\sigma_1 + \sigma_2)^2$ and $(F)_\infty = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2$ results in the deduction that the maximum of F is reached on Γ. The optimality of equally stressed contours is proved.

We note that condition (3.4) is both necessary and sufficient condition for a global optimum.

4. Now, let us assume that there are n holes in a plate which are bounded by the contours Γ_k ($\Gamma = \Sigma\Gamma_k$, $k = 1, 2, \ldots, n$). The boundary conditions on Γ and the conditions at infinity have the form (1.1). In the case under consideration, the application of the property (2.2) permits showing that the minimum of the maximum value of F is achieved on Γ upon compliance with the condition (3.4). Let us note that the shapes of the equally stressed contours satisfying (3.4) have been found in the case of two holes as well as for certain periodic systems of holes in [6]. Furthermore, we note that the proof of the inequality $\Delta F \geqslant 0$ was independent of the connectedness of the domain $G + \Gamma$. Hence, the maximum of F (in the domain $G + \Gamma$) is achieved on Γ even in the presence of n equally stressed holes. Therefore, equally stressed holes will be optimal in the cases under consideration also.

5. Now, let us consider the case when F is given by (1.5), which corresponds to the Tresca plasticity criterion. The boundary values for the stress and the conditions at the infinitely remote point will be assumed as before. It can be shown for the functional under consideration that the boundary values of F are determined on Γ by the same formula $(F)_\Gamma = \sigma_t^2$ as in Sect 3.

Hence, the assertion that the minimum of the maximum value of F is achieved on the contour Γ remains true. The proof of the inequality $\Delta F \geqslant 0$ becomes shorter since $F = Q$ in the case under consideration. We have $\Delta F = 16 (\nabla u)^2 \geqslant 0$. Therefore, equally stressed holes turn out to be optimal even in the sense of the Tresca criterion.

6. Let us examine the more general dependence of F on the stress tensor invariants I_1, I_2 by assuming nonnegativity of the first and second partial derivatives of F with respect to I_2, i.e.,

$$F = F(I_1, I_2), \quad \partial F / \partial I_2 \geqslant 0, \quad \partial^2 F / \partial I_2^2 \geqslant 0 \tag{6.1}$$

On the basis of the properties of homogeneity and positivity of F as well as with the equalities $(I_1)_\Gamma = \sigma_t$, $(I_2)_\Gamma = 0$ taken into account, we arrive at the following expression $(F)_\Gamma = a\sigma_t^2$, where $a > 0$. Use of the representation for $(F)_\Gamma$ and the property of harmonic functions noted in Sect. 2, results directly in the deduction that a minimum of the maximum of F is achieved on Γ for equally stressed contours. Furthermore, let us take account of condition (6.1) and the fact that in the case of equally stressed contours $I_1 = \sigma_1 + \sigma_2$ in the domain $G + \Gamma$. Performing computations completely analogous to those executed in Sect. 3, it can be shown that for a plate with holes satisfying the condition (3.4), the following inequality is valid

$$\Delta F = \frac{d^2F}{dI_2^2}(\nabla I_2)^2 + 4\frac{dF}{dI_2}(\nabla u)^2 \geqslant 0, \quad (x, y) \in G + \Gamma \tag{6.2}$$

Under the assumptions made, the function F turns out to be superharmonic, and therefore, does not reach the maximum at interior points of the domain $G + \Gamma$. Let us compare the values $F(\sigma_1 + \sigma_2, 0)$ and $F(\sigma_1 + \sigma_2, -\sigma_1\sigma_2)$ taken by the functions F on Γ and at the infinitely remote point. Noting that $(I_2)_\infty \leqslant (I_2)_\Gamma$ and using the first of the inequalities (6.1), we obtain $(F)_\Gamma > (F)_\infty$. Therefore, in the case under consideration (3.5) is valid. It hence results that the equally stressed holes are optimal.

7. The analyses performed in Sect. 3 can be extended to the case when a pressure σ_0 ($\sigma_0 \geqslant 0$ is a given constant) is applied to the hole boundaries, i.e., $(\sigma_n)_\Gamma = -\sigma_0$. The second boundary condition on Γ and the conditions at the infinitely remote point are given by (1.1). Using these boundary conditions and (3.1), we obtain an expression for values of the function F on the hole contour $(F)_\Gamma = \sigma_t^2 + \sigma_0\sigma_t + \sigma_0^2$. On the basis of the formula presented for $(F)_\Gamma$ and the property (2.2), it can be shown that the mimimum of the maximum value of $(F)_\Gamma$ is realized if and only if $(\sigma_t)_\Gamma = \sigma_0 + \sigma_1 + \sigma_2$. It is easy to note that the inequalities $(F)_\Gamma > (F)_\infty$, $\Delta F \geqslant 0$, and therefore, the relationship (3.5) also, are valid in this case. Hence, equally stressed holes which satisfy the condition $(\sigma_t)_\Gamma = \sigma_0 + \sigma_1 + \sigma_2$, are optimal.

Problems similar to those considered in this paper occur when seeking the optimal shapes of elastic bodies of finite size in the construction of optimal "conjugates" for bodies with sharply varying geometrics and high stress concentrations and in designing optimal reinforcements of holes [11 - 16].

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REFERENCES

1. Banichuk, N. V., On a variational problem with unknown boundaries and determination of the optimal shapes of elastic bodies. PMM, Vol. 39, No. 6, 1975.

2. Kurshin, L. M., On the problem of determining the shape of a rod section of maximum torsional stiffness. Dokl. Akad. Nauk, SSSR, Vol. 223, No. 3, 1975.

3. Banichuk, N. V. and Karihaloo, B. L., Minimum-weight design of multipurpose cylindrical bars. (English translation), Pergamon Press., Internat. J. Solids and Struc., Vol. 12, No. 4, 1976.

4. Banichuk, N. V., On a two-dimensional optimization problem in the theory of torsion of elastic rods. Izv. Akad. Nauk. SSSR, Mekah. Tverd. Tela, No. 5, 1976.

5. Cherepanov, G. P., Some problems of the theory of elasticity and plasticity with unknown boundaries. In: Application of Function Theory in the Mechanics of a Continuous Medium. Vol. 1, "Nauka", Moscow, 1965.

6. Cherepanov, G. P., Inverse problems of a plane theory of elasticity. PMM, Vol. 38, No. 6, 1974.

7. Ivanov, G. M. and Kosmodamianskii, O. S., Inverse doubly-periodic problem of plane elasticity theory. Dopov. Akad. Nauk. USSR, No. 9, 1972.

8. Kosmodamianskii, A. S., Stress Distribution in Isotropic Multiply-connected Media. Donetsk Univ. Press, 1972.

9. Vladimirov, V. S., Equations of Mathematical Physics. "Nauka", Moscow, 1976.

10. Muskhelishvili, N. I., Some Fundamental Problems of the Mathematical Theory of Elasticity. "Nauka", Moscow, 1966.

11. Hutchinson, J. W., and Niordson, F. I., Designing vibrating membranes. In: Mechanics of a Continuous Medium and Kindred Problems of Analysis. "Nauka", Moscow, 1972.

12. Neuber, H., Der zugbeanspruchte Flachstab mit optimalem Querschnitt - sübergang. Forsch. Geb. Ingenieurwesens., Vol. 35, 1969.

13. Neuber, H., Zur Optimierung der Spannungskonzentration. In: Mechanics of a Continuous Medium and Kindred Problems of Analysis, "Nauka", Moscow, 1972.

14. Tvergaard, V., On the optimum shape of a fillet in a flat bar with restrictions. In: Optimization in Structural Design. Springer- Verlag, Berlin-Heidelberg - New York, 1975, (IUTAM Sympos. Warsaw, Poland, 1973).

15. Aleksandrov, A. Ia., Some solutions of problems about the equivalent and equistrength reinforcements of holes in plates. In: Selected Problems of Applied Mechanics. VINITI, Moscow, 1974.

16. Wheeler, L., On the role of constant-stress surfaces in the problem of minimizing elastic stress concentration. Internat. J. Solids and Struc. Pergamon Press. Vol. 12, No. 11, 1976.

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